Problems for NCUMC 2017. 23.04.2017

1. Do three vectors $\vec{a}, \vec{b}, \vec{c}$ in $\mathbb{R}^3$ exist such that the following three inequalities take place simultaneously:

$$\sqrt{3}|\vec{a}| < |\vec{b} - \vec{c}|, \quad \sqrt{3}|\vec{b}| < |\vec{c} - \vec{a}|, \quad \sqrt{3}|\vec{c}| < |\vec{a} - \vec{b}|?$$

2. Find all non-zero functions $f : \mathbb{C} \to \mathbb{C}$ satisfying the equality $f(x)f(y) = f(x + e^{it}y)$ for fixed $t \in (0, \pi)$. and any $x, y \in \mathbb{C}$.

3. Find the product of all solutions to the equation

$$\sum_{k=1}^{2017} \frac{1}{z - \varepsilon_k} = 0\,,$$

where $\varepsilon_k = e^{\imath k\pi/1009}$ are different zeros of the polynomial $z^{2018} - 1$.

4. Does the following series converge $\sum_{n=1}^{\infty}\{(\sqrt{2}+1)^{2n}\}$? Here $\{a\} = a - [a]$, $[a]$ is the maximal integer less or equal a.

5. Find the maximal set of points in $\mathbb{C}$ such that there are no complex Hermitian positively definite matrices of identical sizes A, B for which the point is an eigenvalue of matrix $(A + B)^{-1}(I + AB)$.

6. Let f be continuous non-negative 2π-periodic function, $0 \le r < 1$. Prove, that

$$\int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos t} f(t)dt \le 2\frac{(1 - r^2) + \pi^2}{1 + r} \int_0^{\infty} \frac{(1 - r)s}{((1 - r)^2 + s^2)^2} \left(\int_{-s}^{s} f(t)dt\right)ds$$

7. Let (A, B, C, D) be a quadraple of four real numbers for which AB, CD, AD, BC are not integers. Determine the convergence of the series

$$\sum_{m=0}^{\infty} m \frac{\binom{AB}{m}\binom{CD}{m}}{\binom{AD-1}{m}\binom{BC-1}{m}}$$

and evaluate its sum when it converges. Here

$$\binom{z}{m} = \frac{\Gamma(z + 1)}{\Gamma(m + 1)\Gamma(z - m + 1)},$$

Γ is the Euler gamma-function.