Problems and solutions for NCUMC 2017. 23.04.2017

1. Do three vectors $\vec{a}, \vec{b}, \vec{c}$ in $\mathbb{R}^3$ exist such that the following three inequalities take place simultaneously:

$$\sqrt{3}|\vec{a}| < |\vec{b} - \vec{c}|, \quad \sqrt{3}|\vec{b}| < |\vec{c} - \vec{a}|, \quad \sqrt{3}|\vec{c}| < |\vec{a} - \vec{b}|?$$

Solution. Let us find squares of each inequality and summarize all the inequalities. One gets

$$3(\vec{a}^2 + \vec{b}^2 + \vec{c}^2) < 2(\vec{a}^2 + \vec{b}^2 + \vec{c}^2) - 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}).$$

Hence, one comes to incorrect inequality

$$(\vec{a} + \vec{b} + \vec{c})^2 < 0.$$

It means that such triplet of vectors does not exist.

2. Find all non-zero functions $f : \mathbb{C} \to \mathbb{C}$ satisfying the equality $f(x)f(y) = f(x + e^{it}y)$ for fixed $t \in (0, \pi)$. and any $x, y \in \mathbb{C}$

Answer. $f(x) = 1$ for all $x \in \mathbb{C}$.

Solution. First we'll show that $f(x) \neq 0$ for all $x \in \mathbb{C}$. If instead there would be z such that $f(z) = 0$, then $f(x) = f(x - \varepsilon z + \varepsilon z) = f(x - \varepsilon z)\, f(z) = 0$ ($\varepsilon = e^{it}$) for all $x \in \mathbb{C}$ which contradicts to the non-nullity of f.

Now, putting $y = \frac{1}{1-\varepsilon}x$ (as $\varepsilon \neq 1$) gives $f(x)f(y) = f(x + \varepsilon y) = f(y)$. Due to $f(y) \neq 0$, one has $f(x) = 1$ for all $x \in \mathbb{C}$.

3. Find the product of all solutions to the equation

$$\sum_{k=1}^{2017} \frac{1}{z-\varepsilon_k} = 0\,,$$

where $\varepsilon_k = e^{\imath k\pi/1009}$ are different zeros of the polynomial $z^{2018}-1$.

Solution. We show it for a general case, i. e. for

$$\sum_{k=1}^{n-1} \frac{1}{z-\varepsilon_k} = 0\,,$$

where ε_k are different zeros of the polynomial $z^n - 1$.

Let us denote $P(z) = \prod\limits_{k=1}^{n-1}(z-\varepsilon_k) = \frac{z^n-1}{z-1}$. Then

$$\sum_{k=1}^{n-1} \frac{1}{z-\varepsilon_k} = \frac{\sum\limits_{k=1}^{n-1}\prod\limits_{j\neq k}(z-\varepsilon_j)}{\prod\limits_{k\neq 0}(z-\varepsilon_k)} = \frac{P'(z)}{P(z)}.$$

Hence the solution to the given equation is the solution to $0 = P'(z) = \frac{(n-1)z^n - nz^{n-1}+1}{(z-1)^2}$, and vice-versa. So the all solutions in question are zeros of the polynomial $(n-1)z^n - nz^{n-1}+1$ without two 1-s (which are the zeros of the denominator). By Vieta's formula the requested product (multiplied by 1^2) is equal to $(-1)^n\frac{1}{n-1}$.

In our case $n = 2018$, so the answer is $\frac{1}{2017}$.

4. Does the following series converge $\sum_{n=1}^{\infty}\{(\sqrt{2}+1)^{2n}\}$? Here $\{a\} = a-[a]$, $[a]$ is the maximal integer less or equal a.

Answer. The series diverges.

Solution. Consider the behavior of the series term. Binomial formula leads to the following two expressions:

$$(1+\sqrt{2})^{2n} = 1+\binom{2n}{1}\cdot\sqrt{2}+\binom{2n}{2}\cdot(\sqrt{2})^2+\binom{2n}{3}\cdot(\sqrt{2})^3+...+\binom{2n}{2n}\cdot(\sqrt{2})^{2n},$$

$$(1-\sqrt{2})^{2n} = 1-\binom{2n}{1}\cdot\sqrt{2}+\binom{2n}{2}\cdot(\sqrt{2})^2-\binom{2n}{3}\cdot(\sqrt{2})^3+...+\binom{2n}{2n}\cdot(\sqrt{2})^{2n}.$$

By summarizing of these two equalities, one obtains

$$(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n} = 2(1+\binom{2n}{2}\cdot(\sqrt{2})^2+\binom{2n}{4}\cdot(\sqrt{2})^4+...+\binom{2n}{2n}\cdot(\sqrt{2})^{2n}) =$$

$$2(1+\binom{2n}{2}\cdot 2+\binom{2n}{4}\cdot 2^2+...+\binom{2n}{2n}\cdot 2^n).$$

It is integer even. Let us mark it as $2A$. Then,

$$(\sqrt{2}+1)^{2n} = 2A-(\sqrt{2}-1)^{2n} = (2A-1)+(1-(\sqrt{2}-1)^{2n}).$$

Hence, $(1+\sqrt{2})^{2n}$ equals a sum of integer $2A-1$ and $1-(\sqrt{2}-1)^{2n}$. The last term satisfies the inequalities $0 < 1-(\sqrt{2}-1)^{2n} < 1$. Consequently,

$$\{(\sqrt{2}+1)^{2n}\} = 1-(\sqrt{2}-1)^{2n}.$$

As $0 < \sqrt{2}-1 < 1$, one has

$$\lim_{n\to\infty}(\sqrt{2}-1)^{2n} = 0.$$

Consequently,

$$\lim_{n\to\infty}\{(\sqrt{2}+1)^{2n}\} = \lim_{n\to\infty}(1-(\sqrt{2}-1)^{2n}) = \lim_{n\to\infty}1-\lim_{n\to\infty}(\sqrt{2}-1)^{2n} = 1.$$

Thus, there is a violation of the necessary condition of convergence for the series. The series diverges.

5. Find the maximal set of points in $\mathbb{C}$ such that there are no complex Hermitian positively definite matrices of identical sizes A, B for which the point is an eigenvalue of matrix $(A+B)^{-1}(I+AB)$.

Solution. Let c be an eigenvalue of the operator in question, i.e. $(A+B)^{-1}(I+AB)x = cx$ for some non-zero vector x and some complex number c. Then,

$$x + ABx = c(Ax + Bx). \tag{1}$$

Mark $Bx = y$. Hence, $(y, x) > 0$, and it is the only condition for x, y. Equation (1) is rewritten in the form $A(y - cx) = cy - x$. Moreover, $(cy - x, y - cx) > 0$ due to the fact that A be positively definite. It is also possible that $y = cx$, $cy = x$. this takes place for $x = y$, $c = 1$. Let us introduce a notation $c = a + b\imath$. Then,

$$a((x,x) + (y,y)) + b\imath((y,y) - (x,x)) - (1 + a^2 + b^2)(x,y) > 0.$$

Consequently, $a > 0$, and for any $a > 0$ and any b, one can find matrices and "almost orthogonal"vectors x, y of identical lengths such that the inequality takes place.

Answer. Left half-plane.

6. Let f be continuous non-negative 2π-periodic function, $0 \le r < 1$. Prove, that

$$\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos t} f(t)dt \le 2\frac{(1-r^2)+\pi^2}{1+r} \int_0^{\infty} \frac{(1-r)s}{((1-r)^2+s^2)^2} (\int_{-s}^{s} f(t)dt)ds$$

Solution. Lemma. For $0 \le r < 1, 0 \le t \le \pi$, one has

$$\frac{(1-r)^2}{1+r^2-2r\cos t} \le \frac{1-r}{(1-r)^2+t^2} \frac{(1-r)^2+\pi^2}{1+r}$$

$$A = \frac{1-r^2}{1+r^2-2r\cos t} = \frac{(1-r)(1+r)}{(1-r)^2+4r\sin^2\frac{t}{2}} = \frac{(1-r)(1+r)}{(1-r)^2+t^2} \frac{(1-r)^2+t^2}{(1-r)^2+4r\sin^2\frac{t}{2}}$$

For $0 \le t \le \pi$ one has $\sin\frac{t}{2} \ge \frac{t}{\pi}$. Correspondingly,

$$A \le \frac{(1-r)(1+r)}{(1-r)^2+t^2} \frac{(1-r)^2+t^2}{(1-r)^2+\frac{4r}{\pi^2}t^2}$$

It is simple to show that for $0 \le r < 1, 0 \le t \le \pi$, the following inequality takes place:

$$B = \frac{(1-r)^2+t^2}{(1-r)^2+\frac{4r}{\pi^2}t^2} \le \frac{(1-r)^2+\pi^2}{(1+r)^2}$$

Really, for $r = 0$, it is evident. Let $0 < r < 1$

$$B = \frac{\pi^2}{4r}(1 - \frac{(1-r)^2(\frac{\pi^2}{4r}-1)}{\frac{\pi^2}{4r}(1-r)^2+t^2}) \le \frac{\pi^2}{4r}(1 - \frac{(1-r)^2(\frac{\pi^2}{4r}-1)}{\frac{\pi^2}{4r}(1-r)^2+\pi^2}) = \frac{(1-r)^2+\pi^2}{(1+r)^2}$$

Here we use the fact, that $\frac{\pi^2}{4r} > 1$. The Lemma is proved.
Due to the Lemma,

$$\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos t} f(t)dt = \int_0^{\pi} \frac{1-r^2}{1+r^2-2r\cos t} (f(t)+f(-t))dt$$

$$\le k\int_0^{\pi} \frac{u}{u^2+t^2}(f(t)+f(-t))dt \le k\int_0^{\infty} \frac{u}{u^2+t^2}(f(t)-f(-t))dt$$

Here $u = 1-r$, $k = \frac{(1-r)^2+\pi^2}{1+r}$. We used the continuity and, correspondingly, boundedness of 2π-periodic function f. Hence, the integral converges. The last integral equals

$$k\int_0^{\infty}(f(t)+f(-t))\int_t^{\infty} \frac{2us}{(u^2+s^2)^2} dsdt = k\int_0^{\infty} \frac{2us}{(u^2+s^2)^2} \int_0^{s}(f(t)+f(-t))dtds$$

$$= 2k\int_0^{\infty} \frac{us}{(u^2+s^2)^2} (\int_{-s}^{s} f(t)dt)ds$$

Here we changed the order of integration. As a result, we come to an expression of the required form.

7. Let (A, B, C, D) be a quadraple of four real numbers for which AB, CD, AD, BC are not integers. Determine the convergence of the series

$$\sum_{m=0}^{\infty} m \frac{\binom{AB}{m}\binom{CD}{m}}{\binom{AD-1}{m}\binom{BC-1}{m}}$$

and evaluate its sum when it converges. Here

$$\binom{z}{m} = \frac{\Gamma(z+1)}{\Gamma(m+1)\Gamma(z-m+1)},$$

Γ is the Euler gamma-function.

Solution. Denote $z = AB$, $x = -AD$, $y = -BC$, then $CD = -xy/z$ and we have

$$m \frac{\binom{z}{m}\binom{xy/z}{m}}{\binom{-x-1}{m}\binom{-y-1}{m}} = h(m-1) - h(m),$$

where

$$h(m) = \frac{z(m+x+1)(m+y+1)}{(z+x)(z+y)} \cdot \frac{\binom{z}{m+1}\binom{xy/z}{m+1}}{\binom{-x-1}{m+1}\binom{-y-1}{m+1}}.$$

So, the partial sum of our series equals $h(0) - h(n)$, and the question reduces to finding the limit of $h(n)$ (when it exists). It is straightforward to check that $h(m)/h(m-1)$ behaves like $1 - ((z+x)(z+y)/z)m^{-1} + O(m^{-2})$ and so $h(n)$ tends to 0 when $(z+x)(z+y)/z > 0$, i.e., $(B-D)(A-C) > 0$, $h(n)$ tends to infinity when $(B-D)(A-C) < 0$. Really,

$$h(m) \sim h(m-1)(1 - \frac{a}{m}) \sim h(m-2)(1 - \frac{a}{m})(1 - \frac{a}{m-1}) \sim ... \sim$$

$$h(0)(1 - \frac{a}{m})(1 - \frac{a}{m-1})...(1-a).$$

Correspondingly,

$$\ln h(m) \sim \ln h(0) + \ln(1 - \frac{a}{m}) + ... + \ln(1-a). \qquad (2)$$

All terms in (2) (besides the first one) have the same sign (negative for $a > 0$ and positive for $a < 0$). To consider the convergence of the series with the partial sum (2),one can use the Gauss theorem.

$$\frac{|\ln(1 - \frac{a}{m})|}{|\ln(1 - \frac{a}{m-1})|} \sim \frac{|a/m|}{|a/(m-1)|} = 1 - \frac{1}{m}.$$

It means that the series with the partial sum (2) diverges to $-\infty$ if $a > 0$ (correspondingly, $h(m) \to 0$) or to $+\infty$ if $a < 0$ (correspondingly, $h(m) \to \infty$).

It is easy to see that the initial series diverges as a harmonic series when $B = D$ or $A = C$. Thus, the series converges to $h(0) = zxy/(z+x)(z+y) = ABCD/(A-C)(B-D)$ when $(B-D)(A-C) > 0$ and diverges otherwise.