2nd NCUMC problems 21.04.2015

1. Let $P(x)=a_0x^{2015}+a_1x^{2014}+...+a_{2015},\ a_0\neq 0,\ a_i\in\{-1,0,1\}, i=0,1,...2015$.

Does the integral $\int_2^\infty \frac{dx}{P(x)}$ converge?

2. Find the limit if it exists: $\lim_{n\to\infty}\underbrace{\sqrt{20\sqrt[3]{15\sqrt{20\sqrt[3]{15\ldots\sqrt{20\sqrt[3]{15}}}}}}}_{2n\ roots}$

3. Calculate M^{100}, where $M=\begin{pmatrix}1 & 2 & 0\\ -3 & -3 & 1\\ 2 & 2 & -1\end{pmatrix}$.

4. Prove inequality $\int_0^{\pi/2}\frac{x}{\sin x}dx\leq\frac{\pi^3}{16}$

5. Let $x(t)$ be a nontrivial solution to the system $\frac{dx}{dt}=Ax$, where $A=\begin{pmatrix}1 & 6 & 2\\ -4 & 4 & 7\\ -2 & -3 & 7\end{pmatrix}$.

Prove that $t\mapsto\|x(\mathrm{t})\|$ is an increasing function ($R\to R$). Here $\|.\|$ denotes the Euclidean norm.

6. Let us say that a parallelepiped in R^3, with edges parallel to coordinate axes, is “semi-integer” if four of its edges, which are parallel to some coordinate axis, has an integer length. Let us compose a parallelepiped from finite number of semi-integer parallelepipeds (above mentioned axis and integer for different small parallelepipeds may be different). Prove that the composed parallelepiped is semi-integer.

7. Let A and B be $n\times n$ Hermitian complex matrices such that the list of all non-zero eigenvalues of $A+B$ counted with respect to multiplicities, is exactly the concatenation of the corresponding lists of non-zero eigenvalues of A and B (possibly after reordering). Show that $AB=0$.