Problem 2. Sequences $\{a_n\}$, $\{b_n\}$ are not convergent, but sequences $\{a_n+b_n\}$, $\{a_nb_n\}$ are convergent. Prove that the sets of condensation points (partial limits) of the sequences $\{a_n\}$, $\{b_n\}$ are the same and consists of two points.

Solution. Let $\lim(a_n+b_n)=a$ and $\lim(a_nb_n)=b$. Subtracting $4b$ from the square of a one obtains $\lim(a_n-b_n)^2=a^2-4b$. It is clear that sequence $\{a_n-b_n\}$ is not convergent. In the opposite case it leads to the convergence of $\{a_n\}$ and $\{b_n\}$ that is wrong. So, we obtain that $a^2-4b>0$ and sequence $\{a_n-b_n\}$ has exactly two condensation points $\pm\sqrt{a^2-4b}$. Taking into account that $\lim(a_n+b_n)=a$, one obtains that the condensation points for $\{a_n\}$ are $(a\pm\sqrt{a^2-4b})/2$ and for $\{b_n\}$ are $(a\mp\sqrt{a^2-4b})/2$, i.e. the sets of the condensation points coincide.