Find the limit if it exists^ $\lim_{n\to\infty}\underbrace{\sqrt{20\sqrt[3]{15\sqrt{20\sqrt[3]{15\ldots\sqrt{20\sqrt[3]{15}}}}}}}_{2n\ roots}$

Solution

Introduce a sequence

$$x_1=\sqrt{20\sqrt[3]{15}}\ ,\ x_n=\underbrace{\sqrt{20\sqrt[3]{15\sqrt{20\sqrt[3]{15\ldots\sqrt{20\sqrt[3]{15}}}}}}}_{2n\ roots}\ \text{ for } n=2,3,\ldots$$

This sequence is

1. bounded from above:

$$x_n=\underbrace{\sqrt{20\sqrt[3]{15\sqrt{20\sqrt[3]{15\ldots\sqrt{20\sqrt[3]{15}}}}}}}_{2n\ roots}<\underbrace{\sqrt{20\sqrt[3]{20\sqrt{20\sqrt[3]{20\ldots\sqrt{20\sqrt[3]{20}}}}}}}_{2n\ roots}<$$

$$<\underbrace{\sqrt{20\sqrt{20\sqrt{20\ldots\sqrt{20}}}}}_{2n\ roots}=20^{\frac{1}{2}+\frac{1}{4}+\ldots+\left(\frac{1}{2}\right)^{2n}}<20^{\frac{1}{2}+\frac{1}{4}+\ldots+\left(\frac{1}{2}\right)^{2n}+\ldots}=20^{\frac{0.5}{1-0.5}}=20.$$

2. non-decreasing:

$$x_{n+1}=\underbrace{\sqrt{20\sqrt[3]{15\sqrt{20\sqrt[3]{15\ldots\sqrt{20\sqrt[3]{15}}}}}}}_{2(n+1)\ roots}=\alpha_n x_n,$$

where $\alpha_n=\left(\sqrt{20\sqrt[3]{15}}\right)^{m_n}\geq 1$, as $\left(\sqrt{20\sqrt[3]{15}}\right)>1$ .

Hence, the limit exists. Let it be $A, A>0$. One has the following equation for $A$

$$A=\sqrt{20\sqrt[3]{15\sqrt{20\sqrt[3]{15\sqrt{20\sqrt[3]{15\ldots}}}}}}=\sqrt{20\sqrt[3]{15A}},$$

которое преобразуем к виду $A^6=20^3\,15\,A$. The answer is given by the non-zero root $A=\left(20^3\,15\right)^{1/5}$.