Problem 4. Find all functions $f(x) : (0, +\infty) \to (0, +\infty)$ satisfying

$$\frac{1}{1 + x + f(y)} + \frac{1}{1 + y + f(z)} + \frac{1}{1 + z + f(x)} = 1$$

whenever x, y, z are positive numbers and $xyz = 1$.

Solution. It is straightforward to ensure that $f(x) = 1/x$ satisfies this condition. Let's check that this the unique answer. Substitute $x = y = z = 1$ and get $f(1) = 1$. Substitute $z = 1$, $x = t$, $y = 1/t$, where $t \neq 1$. Denoting $f(1/t) = A$, $f(t) = B$ we get $\frac{1}{1+t+A} + \frac{1}{2+1/t} + \frac{1}{2+B} = 1$, or equivalently $AB(t+1) + A + Bt^2 = 3t + 1$. Replacing t and $1/t$ we get $AB(1/t + 1) + B + A/t^2 = 3/t + 1$, multiplying by t^2 it transforms into $AB(t + t^2) + Bt^2 + A = 3t + t^2$, subtracting we get $AB(t^2 - 1) = t^2 - 1$, $AB = 1$, next $A + t^2/A = A + Bt^2 = 3t + 1 - AB(t + 1) = 2t$, hence $f(1/t) = A = t$ for all t as desired.