Problem 6. Prove that there exists integer r such that first hundred of digits of the number e^r coincide with the first hundred of digits of the number π.

Solution. Prove that if $\lg a = \log_{10} a$ is irrational then for any set of digits $b_1,...b_n$ one can find such integer x that first n digits of a^x coincide with $b_1,...b_n$. Note that one can assume that $a>1$.

Introduce the notation $b = b_1 \cdot 10^{n-1} + ... + b_n$. Then, first n digits of a^x coincide with b if there is natural y satisfying the condition:

$b \cdot 10^y \le a^x < (b+1)10^y$.

Then,

$\lg b + y \le x \lg a < y + \lg(b+1)$

As $\lg a$ is irrational then the continuous fractions theory allows us to conclude that there is infinite set of integers q and p such that

$$0 < q \lg a - p < \frac{1}{q}$$

Let us take q so large that $\frac{1}{q} < \lg(b+1) - \lg b$. Hence, $0 < q \lg a - p < \lg(b+1) - \lg b$.

Consequently, there exists integer m such that

$mq \lg a - mp \in [\lg b, \lg(b+1)]$

Taking $y = mp$, $x = mq$, One obtains the needed statement.

If one choose as $b_1,.... b_{100}$ first 100 digits of π then the proper value of integer r is $r = x$, due to irrationality of $\lg e$. This completes the proof.

Addition. One can use the Kronecker theorem and obtain more general result. If $\lg a_1, ...\lg a_k, 1$ $a_i, ..., a_k > 1$ are linearly independent over Z, then for any sets of digits $b_1 = b_{11}, ..., b_{1m_1}, ...b_k = b_{k1}...b_{km_k}$ one can find integer n such that numbers $a_1^n, ..., a_k^n$ have $b_1, ..., b_k$ as the first digits, correspondingly.

For example, there is such integer n that $3^n, 7^n, 11^n$ has the first 100 digits coinciding with that of $e, \pi, \sqrt{3}$, correspondingly.