6. Let us say that a parallelepiped in R^3, with edges parallel to coordinate axes, is "semi-integer" if four of its edges, which are parallel to some coordinate axis, has an integer length. Let us compose a parallelepiped from finite number of semi-integer parallelepipeds (above mentioned axis and integer for different small parallelepipeds may be different). Prove that the composed parallelepiped is semi-integer.

Solution.

Lemma. Let $a_1 < a_2$. Then the integral $\int_{a_1}^{a_2} e^{2\pi ix} dx$ is zero if and only if $a_2 - a_1 \in \mathbb{Z}$.

Proof of the Lemma.

$$\int_{a_1}^{a_2} e^{2\pi ix} dx = \frac{1}{2\pi i}(e^{2\pi ia_2} - e^{2\pi ia_1}) = \frac{e^{2\pi ia_1}}{2\pi i}(e^{2\pi i(a_2-a_1)} - 1).$$

It gives the statement of the Lemma.

Let Q be the parallelepiped

$$Q = \{(x,y,z) : a_1 \le x \le a_2, \mathrm{b}_1 \le y \le b_2, \mathrm{c}_1 \le z \le c_2\}.$$

Consider

$$I(Q) = \iiint_Q e^{2\pi i(x+y+z)} dxdydz = \int_{a_1}^{a_2} e^{2\pi ix} dx \int_{b_1}^{b_2} e^{2\pi iy} dy \int_{c_1}^{c_2} e^{2\pi iz} dz.$$

In accordance with the Lemma $I(Q) = 0$ if and only if at least one of the numbers $a_2 - a_1, b_2 - b_1, c_2 - c_1$ is integer.

Let $Q = \bigcup_{j=1}^{m} Q_j$, where $Q_j, j = 1,2,...m,$ are semi-integer. Then

$$I(Q) = \sum_{j=1}^{m} I(Q_j) = \sum_{j=1}^{m} 0 = 0.$$

This means that Q is semi-integer.