Problem 7. Let $q(x)$ is bounded from below, i.e. there exists constant $c>0$ such that $q(x)>-c$ for all x, and $\lim_{x\to\infty}\int_x^{x+\omega} q(x)dx=\infty$ for any $\omega>0$. Prove that for any fixed λ any non-trivial solution of the equation $y''+\left(\lambda-q(x)\right)y=0$ has finite number of roots at $(0,\infty)$.

Solution. One can see that adding of a constant does not change the situation. Hence, it is sufficient to consider the case $q(x)\geq 0$. Let there exist $\lambda=\lambda_0>0$ such that there is a solution $y=y(x)$ of the equation $y''+\left(\lambda-q(x)\right)y=0$ having infinite number of roots $\alpha_1<\alpha_2<\alpha_3<...<\alpha_n<...$ Evidently that for $\lambda=\lambda_0<0$ it is not possible due to the sign of $y'', y''=-\left(\lambda-q(x)\right)y$. Really, if at some root α one has $y'(\alpha)>0$ then for $x>\alpha$ one has $y(x)>0$ and, consequently, $y''>0$ and $y'>0$. Hence, there is no other roots greater than α.

Let ω be such (small) positive number that $\omega<\frac{1}{\lambda_0+1}$. Let us choose N so large that for $x>N$ one has $\int_x^{x+\omega} q(t)dt>\omega(\lambda_0+1)$ (1).

It is possible due to the condition for $q(x)$.

Note that the set of the solution roots has no accumulation points. Hence, we can choose n in such a way that $\alpha_n>N$, and, later, choose $m,\ m>n,$ such that $\alpha_m-\alpha_n>\omega$. If one takes $\omega_1=P\omega,\ P-$ positive integer, then $\int_x^{x+P\omega} q(t)dt>\int_x^{x+\omega} q(t)dt>\omega(\lambda_0+1)$ .due to positivity of q for the same x. Hence, we can believe that $\alpha_m-\alpha_n=P\omega$.

Let us rewrite the equation $y''+\left(\lambda-q(x)\right)y=0$ for $\lambda=\lambda_0$ in the form $y''=\left(q(x)-\lambda_0\right)y$.

Let us multiply the both parts by y and integrate from α_n to α_m. Integration by parts of the left hand side leads to the equation

$$-\int_{\alpha_n}^{\alpha_m}\left(y'(t)\right)^2dt=\int_{\alpha_n}^{\alpha_m} q(t)y^2(t)dt-\lambda_0\int_{\alpha_n}^{\alpha_m} y^2(t)dt. \qquad (2)$$

The first integral in the right hand side can be represented in the form

$$\int_{\alpha_n}^{\alpha_m} q(t)y^2(t)dt=\sum_{k=1}^{P}\int_{\alpha_n+(k-1)\omega}^{\alpha_n+k\omega} q(t)y^2(t)dt.$$

The mean value theorem and (1) leads to the following inequality

$$\int_{\alpha_n+(k-1)\omega}^{\alpha_n+k\omega} q(t)y^2(t)dt>(\lambda_0+1)y^2(\xi_k)\omega,$$

$$\alpha_n+(k-1)\omega<\xi_k<\alpha_n+k\omega.$$

Hence,

$$\int_{\alpha_n}^{\alpha_m} q(t)y^2(t)dt>(\lambda_0+1)\sum_{k=1}^{P}y^2(\xi_k)\omega=(\lambda_0+1)\int_{\alpha_n}^{\alpha_m} y^2(t)dt-(\lambda_0+1)\sum_{k=1}^{P}\int_{\alpha_n+(k-1)\omega}^{\alpha_m+k\omega}\left(y^2(t)-y^2(\xi_k)\right)dt.$$

(3)

One can obtain the following correlation

$$\left|y^2(t)-y^2(\xi_k)\right| = 2\left|\int_{\xi_k}^{t} y'(u)y(u)du\right| \le \int_{\xi_k}^{t} y^2(u)du + \int_{\xi_k}^{t} \left[y'(u)\right]^2 du \le$$

$$\le \int_{\alpha_{n+(k-1)\omega}}^{\alpha_n+k\omega} y^2(u)du + \int_{\alpha_{n+(k-1)\omega}}^{\alpha_n+k\omega} \left[y'(u)\right]^2 du .$$

Hence, inequality (3) gives us

$$\int_{\alpha_n}^{\alpha_m} q(t)y^2(t)dt > (\lambda_0+1)\int_{\alpha_n}^{\alpha_m} y^2(t)dt - (\lambda_0+1)\sum_{k=1}^{P}\int_{\alpha_n+(k-1)\omega}^{\alpha_n+k\omega}\left(\int_{\alpha_n+(k-1)\omega}^{\alpha_n+k\omega} y^2(u)du\right)dt -$$

$$-(\lambda_0+1)\sum_{k=1}^{P}\int_{\alpha_n+(k-1)\omega}^{\alpha_n+k\omega}\left(\int_{\alpha_n+(k-1)\omega}^{\alpha_n+k\omega}\left[y'(u)\right]^2 du\right)dt =$$

$$= (\lambda_0+1)\int_{\alpha_n}^{\alpha_m} y^2(t)dt - (\lambda_0+1)\omega\int_{\alpha_n}^{\alpha_m} y^2(t)dt - (\lambda_0+1)\omega\int_{\alpha_n}^{\alpha_m}\left[y'(t)\right]^2 dt . \qquad (4)$$

One obtains from (2) and (4)

$$-\int_{\alpha_n}^{\alpha_m}\left[y'(t)\right]^2 dt > \left[1-(\lambda_0+1)\omega\right]\int_{\alpha_n}^{\alpha_m} y^2(t)dt - (\lambda_0+1)\omega\int_{\alpha_n}^{\alpha_m}\left[y'(t)\right]^2 dt ,$$

i.e..

$$\left[1-(\lambda_0+1)\omega\right]\int_{\alpha_n}^{\alpha_m}\left\{y^2(t)+\left[y'(t)\right]^2\right\}dt < 0 ,$$

But it is impossible due to the inequality $(\lambda_0+1)\omega < 1$.