The 70th William Lowell Putnam Mathematical Competition

6 December 2009

Exam A

A1. Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points P in the plane.

A2. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \qquad f(0)=1, \\ g' &= fg^2h + \frac{4}{fh}, \qquad g(0)=1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \qquad h(0)=1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

A3. Let d_n be the determinant of the $n\times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1,\ \cos 2,\ \ldots,\ \cos n^2$. (For example, $d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$. The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

A4. Let S be a set of rational numbers such that

(a) $0\in S$;

(b) If $x\in S$ then $x+1\in S$ and $x-1\in S$; and

(c) If $x\in S$ and $x\notin\{0,1\}$, then $\frac{1}{x(x-1)}\in S$.

Must S contain all rational numbers?

A5. Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2009}?

A6. Let $f:[0,1]^2\to\mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a=\int_0^1 f(0,y)\mathrm{d}y$, $b=\int_0^1 f(1,y)\mathrm{d}y$, $c=\int_0^1 f(x,0)\mathrm{d}x$, and $d=\int_0^1 f(x,1)\mathrm{d}x$. Prove or disprove: There must be a point (x_0,y_0) in $(0,1)^2$ such that $\frac{\partial f}{\partial x}(x_0,y_0)=b-a$ and $\frac{\partial f}{\partial y}(x_0,y_0)=d-c$.