The 70th William Lowell Putnam Mathematical Competition

6 December 2009

Exam B

B1. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example, $\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$.

B2. A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

B3. Call a subset S of $\{1, 2, \ldots, n\}$ *mediocre* if it has the following property: Whenever a and b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1, 2, \ldots, n\}$. [For instance, every subset of $\{1, 2, 3\}$ except $\{1, 3\}$ is mediocre, so $A(3) = 7$.] Find all positive integers n such that $A(n+2) - 2A(n+1) + A(n) = 1$.

B4. Say that a polynomial with real coefficients in two variable, x, y, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.

B5. Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - (f(x))^2}{x^2 \left((f(x))^2 + 1 \right)} \quad \text{for all } x > 1.$$

Prove that $\lim_{x \to \infty} f(x) = \infty$.

B6. Prove that for every positive integer n, there is a sequence of integers $a_0, a_1, \ldots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonnegative integer k, or of the form $b \bmod c$ for some earlier positive terms b and c. [Here $b \bmod c$ denotes the remainder when b is divided by c, so $0 \leq (b \bmod c) < c$.]