The 74th William Lowell Putnam Mathematical Competition
Sunday, 8 December 2013
Exam A

A1 Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegetive integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

A2 Let S be the set of all positive integers that are *not* perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 3 \cdot 4, 2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

A3 Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^n} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

A4 A finite collection of digits 0 and 1 is written around a circle. An *arc* of length $L \geq 0$ consists of L consecutive digits around the circle. For each arc w, let $Z(w)$ and $N(w)$ denote the number of 0's in w and the number of 1's in w, respectively. Assume that $|Z(w) - Z(w')| \leq 1$ for any two arcs w, w' of the same length. Suppose that some arcs $w_1, \ldots, w_k$ have the property that

$$Z = \frac{1}{k}\sum_{j=1}^{k} Z_(w_j) \text{ and } N = \frac{1}{k}\sum_{j=1}^{k} N(w_j)$$

are both integers. Prove that there exists an arc w with $Z(w) = Z$ and $N(w) = N$.

A5 For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} $(1 \leq i < j < k \leq m)$ is said to be *area definite* for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \mathrm{Area}(\triangle A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four number $a_{123} = a_{124} = a_{134} = 1, a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

A6 Define a function $w : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as follows. For $|a|, |b| \leq 2$, let $w(a, b)$ be as in the table shown; otherwise, let $w(a, b) = 0$.

		b				
	$w(a,b)$	-2	-1	0	1	2
	-2	-1	-2	2	-2	-1
	-1	-2	4	-4	4	-2
a	0	2	-4	12	-4	2
	1	-2	4	-4	4	-2
	2	-1	-2	2	-2	-1

For every finite subset S of $\mathbb{Z} \times \mathbb{Z}$, define

$$A(S) = \sum_{(\mathbf{s}, \mathbf{s}') \in S \times S} w(\mathbf{s} - \mathbf{s}').$$

Prove that if S is any finite nonempty subset of $\mathbb{Z} \times \mathbb{Z}$, then $A(S) > 0$. (For example, if $S = \{(0,1), (0.2), (2,0), (3,1)\}$, then the terms in $A(S)$ are $12, 12, 12, 12, 4, 4, 0, 0, 0, 0, -1, -1, -2, -2, -4, -4$.)

The web site of Putnam Competition in Ukraine is `http://putnam.ho.ua/putnam.html`
The problems are CONFIDENTIAL till Monday, 9 December 2013