The 74th William Lowell Putnam Mathematical Competition
Sunday, 8 December 2013
Exam B

B1 For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1, c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

B2 Let $C = \bigcup_{N=1}^{\infty} C_N$, where C_N denotes the set of 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$$

for which:

(i) $f(x) \geq 0$ for all real x, and

(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

B3 Let P be a nonempty collections of subsets of $\{1, \ldots, n\}$ such that:

(i) if $S, S' \in P$, then $S \cup S' \in P$ and $S \cap S' \in P$, and

(ii) if $S \in P$ and $S \neq \emptyset$, then there is a subset $T \in S$ such that $T \in P$ and T contains exactly one fewer element than S.

Suppose that $f : P \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and

$$f(S \cup S') = f(S) + f(S') - f(S \cap S') \text{ for all } S, S' \in P.$$

Must there exist real numbers $f_1, \ldots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i$$

for every $S \in P$?

B4 For any continuous real-valued function f defined on the interval $[0, 1]$, let

$$\mu(f) = \int_0^1 f(x)\, dx, \operatorname{Var}(f) = \int_0^1 (f(x) - \mu(f))^2\, dx, M(f) = \max_{0 \leq x \leq 1} |f(x)|.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0, 1]$, then

$$\operatorname{Var}(fg) \leq 2\operatorname{Var}(f)M(g)^2 + 2\operatorname{Var}(g)M * f)^2.$$

B5 Let $X = \{1, 2, \ldots, n\}$, and let $k \in X$. Show that there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \geq 0$ such that $f^{(j)}(x) \leq k$.

[Here $f^{(j)}$ denotes that jth iterate of f, so that $f^{(0)}(x) = x$ and $f^{(j+1)}(x) = f\left(f^{(j)}(x)\right)$.]

B6 Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?