Problems for NCUMC 2018. 22.04.2018

Problem 1. Any nonnegative polynomial of two real variables reaches its infimum at some point. Is this statement correct?

Problem 2. Let

$$\cos A := I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \frac{1}{6!}A^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} A^{2n},$$

for any square matrix A, where I is the identity matrix. Does there exist a 2×2 square matrix M such that

$$\cos M = \begin{pmatrix} 0 & 2018 \\ 0 & 0 \end{pmatrix}?$$

Problem 3. Let y be real n times continuously differentiable function vanishing outside some finite interval belonging to $(0, \infty)$. Prove the inequality:

$$\int_0^\infty \frac{y^2}{x^{2n}} dx \le \frac{2^{2n}}{((2n-1)!!)^2} \int_0^\infty (y^{(n)})^2 dx.$$

Problem 4. Find all functions $f \in C^2(\mathbb{R}_+)$ such that for any $a \ge 0$:

$$\int_0^a dx \int_0^x f(\frac{ay}{x}) dy = \frac{a}{4}(f(a) + f'(a)), \quad f(0) = 1.$$

Problem 5. Let us consider the set of real orthogonal matrices $O(n, \mathbb{R})$ as a subset of an euclidean space $\mathbb{R}^{n^2}$. It is known that $O(n, \mathbb{R})$ has two components, O_+ contained matrices of determinant equal to 1, and O_- of those which determinant is equal to -1. Compute the euclidean distance between O_+ and O_-.

Remark: The euclidean distance of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ is equal to $\mathrm{dist}(A, B) = \sqrt{\sum_{i,j} |a_{i,j} - b_{i,j}|^2}$.

Problem 6. Let F be locally integrable $2\pi-$periodic function such that

$$\|F\|_* = \sup_I \frac{1}{|I|} \int_I |F(t) - F_I| dt < \infty.$$

Here $F_I = \frac{1}{|I|} \int_I F(t)dt$, $|I|$ is the length of interval I. Consider two intervals I and J with the same middle point, $I \subset J$. Prove that

$$|F_I - F_J| \le 2\left(\log_2 \frac{|J|}{|I|} + 1\right) \|F\|_*.$$

Problem 7. For which natural n the equation

$$y^{(n)}(x) = y^2(x)$$

has a positive solution defined on a semi-axis $(a, +\infty)$ for some a?