Problems and solutions for NCUMC 2018. 22.04.2018

Problem 1. Any nonnegative polynomial of two real variables reaches its infimum at some point. Is this statement correct?

Solution. No. Example $P(x,y) = x^2 + (xy-1)^2$.

$\inf(x^2 + (xy-1)^2) = 0$. Really, one can consider point at the curve $xy = 1$ for $y \to \infty \Rightarrow P(\frac{1}{y}, y) \to 0$. From the other hand, $P(x,y) \neq 0$ everywhere.

Problem 2. Let

$$\cos A := I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \frac{1}{6!}A^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} A^{2n},$$

for any square matrix A, where I is the identity matrix. Does there exist a 2×2 square matrix M such that

$$\cos M = \begin{pmatrix} 0 & 2018 \\ 0 & 0 \end{pmatrix} ?$$

Solution. The answer is positive. For instance

$$\cos \begin{pmatrix} \cdot\frac{\pi}{2} & -2018 \\ 0 & \cdot\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & 2018 \\ 0 & 0 \end{pmatrix}.$$

To prove it, we should note that

$$\begin{pmatrix} \omega & 1 \\ 0 & \omega \end{pmatrix}^n = \begin{pmatrix} \omega^n & n\omega^{n-1} \\ 0 & \omega^n \end{pmatrix}$$

for $n = 0, 1, \ldots$. Hence

$$\cos\left(\beta \begin{pmatrix} \omega & 1 \\ 0 & \omega \end{pmatrix}\right) = \begin{pmatrix} \cos(\beta\omega) & -\beta\sin(\beta\omega) \\ 0 & \cos(\beta\omega) \end{pmatrix}$$

So it is enough to find β and ω such that $\cos(\beta\omega) = 0$ and $-\beta\,\sin(\beta\omega) = 2018$, for instance $\beta = -2018$ and $\beta\omega = \frac{\pi}{2}$.

Problem 3. Let y be real n times continuously differentiable function vanishing outside some finite interval belonging to $(0, \infty)$. Prove the inequality:

$$\int_0^\infty \frac{y^2}{x^{2n}} dx \leq \frac{2^{2n}}{((2n-1)!!)^2} \int_0^\infty (y^{(n)})^2 dx.$$

Solution. Let us present the integral in the left hand side of the inequality in the following form:

$$I = \int_0^\infty \frac{y^2}{x^{2n}} dx = 2 \int_0^\infty x^{-2n} dx \int_0^x y(t)y'(t)dt =$$

$$2 \int_0^\infty y(t)y'(t)dt \int_1^\infty x^{-2n} dx = \frac{2}{2n-1} \int_0^\infty t^{1-2n} y(t)y'(t)dt.$$

Due to Cauchy inequality, one has

$$\int_0^\infty \frac{y^2}{x^{2n}} dx \leq \frac{2}{2n-1} \left(\int_0^\infty \frac{y^2}{x^{2n}} dx \right)^{\frac{1}{2}} \left(\int_0^\infty \frac{(y')^2}{x^{2-2n}} dx \right)^{\frac{1}{2}}.$$

Correspondingly,

$$\int_0^\infty \frac{y^2}{x^{2n}} dx \leq \frac{2^2}{(2n-1)^2} \int_0^\infty \frac{(y')^2}{x^{2n-2}} dx.$$

Let us repeat the procedure for the integral in the right hand side, and then repeat again and again, totally, n times. As a result, we come to the desired inequality.

Problem 4. Find all functions $f \in C^2(\mathbb{R}_+)$ such that for any $a \geq 0$:

$$\int_0^a dx \int_0^x f(\frac{ay}{x})dy = \frac{a}{4}(f(a) + f'(a)), \quad f(0) = 1.$$

Solution.

$$I(a) = \int_0^{\frac{\pi}{4}} f(a\tan\varphi)d\varphi \int_0^{\frac{a}{\cos\varphi}} rdr = \frac{1}{2}\int_0^{\frac{\pi}{4}} f(a\tan\varphi)(\cos\varphi)^{-2}d\varphi = |t = a\tan\varphi| =$$

$$= \frac{a}{2}\int_0^a f(t)dt = \frac{a}{4}(f(a) + f'(a));$$

$$\int_0^a f(t)dt = \frac{1}{2}(f(a) + f'(a)), \quad f(0) = 1 \Rightarrow f'(0) = -1$$

$$f(a) = \frac{1}{2}(f'(a) + f"(a)), \quad f(0) = 1, \quad f'(0) = -1$$

$$f(a) = Ae^x + Be^{-2x}$$

Due to the initial conditions, one has

$$A + B = 1, \quad A - 2B = -1,$$

$$A = \frac{1}{3}, \quad B = \frac{2}{3}.$$

Simple substitution shows that $f(x) = \frac{1}{3}e^x + \frac{2}{3}e^{-2x}$ satisfies the proper relation.
Answer: $f(x) = \frac{1}{3}e^x + \frac{2}{3}e^{-2x}$.

Problem 5. Let us consider the set of real orthogonal matrices $O(n,\mathbb{R})$ as a subset of an euclidean space $\mathbb{R}^{n^2}$. It is known that $O(n,\mathbb{R})$ has two components, O_+ contained matrices of determinant equal to 1, and O_- of those which determinant is equal to -1. Compute the euclidean distance between O_+ and O_-.
Remark: The euclidean distance of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ is equal to $\mathrm{dist}(A,B) = \sqrt{\sum_{i,j} |a_{i,j} - b_{i,j}|^2}$.

Solution. The euclidean distance between any two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is equal to

$$\mathrm{dist}(A,B) = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - b_{ij})^2\Big)^{1/2} = \Big(\mathrm{Tr}\big((A-B)^T(A-B)\big)\Big)^{1/2},$$

where X^T is the transpose of a matrix X. It is induced by the Frobenius norm $\|A\| = \sqrt{\mathrm{Tr}(A^TA)}$. Moreover, for orthogonal matrices A and B we have

$$\mathrm{Tr}\big((A-B)^T(A-B)\big) = \mathrm{Tr}(A^TA - A^TB - B^TA + B^TB) = 2n - \mathrm{Tr}\big(A^TB + (A^TB)^T\big)$$

as $A^TA = B^TB = I$, ie. the identity matrix. If $A \in O_+$ and $B \in O_-$, then $\det(A^TB) = \det(A)\det(B) = -1$. So $A^TB \in O_-$. Of course $I \in O_+$. Hence

$$\mathrm{dist}(O_+,O_-)^2 = \min_{A\in O_+ \& B\in O_-} \mathrm{Tr}\big((A-B)^T(A-B)\big) = 2n - \max_{X\in O_-} \mathrm{Tr}(X + X^T)\,.$$

Any orthogonal matrix X can be brought to the canonical form $U^T\Lambda U$, where

$$\Lambda = \begin{pmatrix} V_1 & & & & & \\ & \ddots & & & 0 & \\ & & V_k & & & \\ & & & \pm 1 & & \\ & 0 & & & \ddots & \\ & & & & & \pm 1 \end{pmatrix}$$

and V_i are 2×2 rotation matrices (with conjugate eigenvalues). So

$$\max_{X\in O_-} \mathrm{Tr}(X + X^T) = \max_{\prod \lambda_j = -1} 2\sum_{k=1}^{n} \mathrm{Re}\,\lambda_k = 2(n-2)\,.$$

Finally, we get $\mathrm{dist}(O_+,O_-)^2 = 2n - 2(n-2) = 4$, that is $\mathrm{dist}(O_+,O_-) = 2$.

Problem 6. Let F be locally integrable $2\pi-$periodic function such that

$$\|F\|_* = \sup_I \frac{1}{|I|} \int_I |F(t) - F_I| dt < \infty.$$

Here $F_I = \frac{1}{|I|} \int_I F(t)dt$, $|I|$ is the length of interval I. Consider two intervals I and J with the same middle point, $I \subset J$. Prove that

$$|F_I - F_J| \leq 2 \left(\log_2 \frac{|J|}{|I|} + 1 \right) \|F\|_*. \tag{1}$$

Solution. First, consider the case when $|I| < |J| \leq 2|I|$. Then,

$$|F_I - F_J| = \frac{1}{|I|} \left| \int_I (F(t) - F_J) dt \right| \leq$$

$$\frac{1}{|I|} \int_I |(F(t) - F_J)| \, dt \leq \frac{2}{|J|} \int_I |(F(t) - F_J)| \, dt \leq 2\|F\|_*.$$

Hence, for the first case inequality (1) is valid. Here we used only that $I \subset J$ and $|I| < |J| \leq 2|I|$.

We will prove the general statement by induction. Let $2^n|I| < |J| \leq 2^{n+1}|I|$ and the statement have been proved for intervals J' such that $|I| < |J'| \leq 2|I|$. Let us take as J' the interval with the same middle point and with two times smaller length. Then, $I \subset J' \subset J$. Due to the induction hypothesis,

$$|F_I - F_{J'}| \leq 2 \left(\log_2 \frac{|J'|}{|I|} + 1 \right) \|F\|_*.$$

As $|J| = 2|J'|$ and the induction base, $|F_I - F_{J'}| \leq 2\|F\|_*$. Consequently,

$$|F_I - F_J| \leq |F_I - F_{J'}| + |F_J - F_{J'}| \leq$$

$$2\|F\|_* + 2\|F\|_* + \left(\log_2 \frac{|J'|}{|I|} + 1 \right) =$$

$$2\|F\|_* \left(\log_2 \frac{|J|}{|I|} + 1 \right).$$

Hence, the statement is valid for $2^n\|I\| < \|J\| < 2^{n+1}$. This finishes the proof (by induction).

Problem 7. For which natural n the equation

$$y^{(n)}(x) = y^2(x) \tag{1}$$

has a positive solution defined on a semi-axis $(a, +\infty)$ for some a?

Solution.

1. For <u>even</u> n such a solution can be defined explicitly:

$$y(x) = Cx^{-n}, \ \text{ where } C = n(n+1)(n+2)...(2n-1), \ \ x \in (0, +\infty).$$

2. Suppose n is <u>odd</u> and $y(x)$ is a positive solution to Eq. (1) defined on a semi-axis $(a, +\infty)$. According to Eq. (1) the function $y^{(n)}(x)$ is also positive and makes the function $y^{(n-1)}(x)$ to strictly increase and therefore to have an eventually constant non-zero sign.

Similarly we obtain eventual strict monotony and non-zero constant sign for $y^{(n-2)}(x), \ldots, y'(x), y(x)$. Thus, all $y(x), y'(x), \ldots, y^{(n)}(x)$ have finite or infinite limits as $x \to +\infty$.

If all these limits equal zero, then the positive and eventually monotone function $y(x)$ must eventually decrease. Hence $y'(x)$ is eventually negative and tends to zero eventually increasing. By the same arguments, all derivatives $y^{(j)}(x), j = 0, \ldots, n$, are eventually positive for even j and negative for odd ones. This contradicts Eq. (1) with odd n.

So, at least one of the above limits, say $\lim\limits_{x\to+\infty} y^{(j)}(x)$, is non-zero. Hence, all derivatives of lower order also have non-zero limits. This holds for $y(x)$ itself, which must have a positive limit, and, according to Eq. (1), for $y^{(n)}(x)$, too. Thus, all $y^{(j)}(x), j = 0, \ldots, n$, must tend to $+\infty$ providing the existence of a point $b > a$ such that $y^{(j)}(b) > 1$ for all $j = 0, \ldots, n$.

Note that the function $z(x) = C(-x)^{-n}$ with the above constant C is a solution to Eq. (1) on $(-\infty, 0)$ regardless of odd or even n. Since all derivatives $z^{(j)}(x), j = 0, \ldots, n$, tend to zero as $x \to -\infty$, there exists a point $-c < 0$ such that $z^{(j)}(-c) < 1$ for all $j = 0, \ldots, n$.

The function $u(x) = z(x - b - c), \ x \in (-\infty, b + c)$, is also a solution to Eq. (1) and satisfies the conditions $u^{(j)}(b) < 1 < y^{(j)}(b)$ for all $j = 0, \ldots, n$.

Now we prove the inequalities $y^{(j)}(x) > u^{(j)}(x)$ for all $x \in (b, b + c)$ and $j = 0, \ldots, n-1$. Suppose $s \in (b, b + c)$ is the most left of the points where at least one of the above inequalities does not hold, say, $y^{(j)}(s) = u^{(j)}(s)$.

According to this selection, the inequality $y^{(j+1)}(x) > u^{(j+1)}(x)$ holds for all $x \in [b, s)$. (This inequality holds for $j = n-1$ as well since $y^{(n)}(x) = y(x)^2 > u(x)^2 = u^{(n)}(x)$ whenever $b \le x < s$.) Integrating this inequality over $[b, s]$ we obtain $y^{(j)}(s) > u^{(j)}(s)$ in contradiction with s selected.

So, $y(x) > u(x)$ whenever $x \in [b, b + c)$. Since $u(x) \to +\infty$ as $x \to b + c$, the solution $y(x)$ cannot be defined on $(a, +\infty) \supset [b, b + c)$.